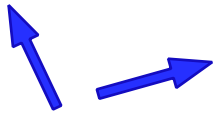
CS770/870 Spring 2017 Quaternions

• Primary resources used in preparing these notes:

1. van Osten, *3D Game Engine Programming: Understanding Quaternions*, https://www.3dgep.com/understanding-quaternions
2. Barbic, *Quaternions and Rotations*, http://run.usc.edu/cs520-s12/quaternions/quaternions-cs520.pdf
3. *OpenGL Tutorials: Tutorial 17: Rotations*, http://www.opengl-tutorial.org/intermediate-tutorials/tutorial-17-quaternions/
4. Ibanez, *Tutorial on Quaternions: Part I*, https://itk.org/CourseWare/Training/QuaternionsI.pdf
5. Dam et al. *Quaternions, Interpolation and Animation*, http://web.mit.edu/2.998/www/QuaternionReport1.pdf

Rotation Specification Problem

- Objects in space have *states* that are changed by operations
  - *location* changed by translation
  - *size* changed by scale
  - *orientation* changed by rotation
- Given start/end *location* and *size*, trivial to determine parameters for translation and scale
- What about *orientation*?
  - In 3D it's especially hard



Remember this!

Rotation about Arbitrary Axis

- Rotate α degrees about an axis defined by a point (c_x,c_y,c_z) and a *direction,* $d=(d_x,d_y,d_z)$
  - Translate (c_x,c_y,c_z) to origin: $T=T(-c_x,-c_y,-c_z)$

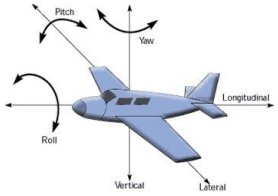
Key steps

  - Rotate about x so that (d_x,d_y,d_z) lies in xy plane
  - Rotate about z so that d' (transformed d) lies along y axis
  - Rotate α degrees about y axis: $R_y(\alpha)$
  - Apply inverse of z rotation
  - Apply inverse of x rotation
  - Apply inverse of translation: $T^{-1}=T(c_x,c_y,c_z)$

Composite transformation: $R = T^{-1}R_x^{-1}R_z^{-1}R_y(\alpha)R_zR_xT$

Rotation Specification Options

- Matrices (as in previous slide)
  - comprehensive, but tedious
- Euler angles (rotations about x, y, z axes)
  - *pitch* (x), *yaw* (y), and *roll* (z) angles
  - Compose in what order?
    - unique: xyz, xzy, yxz, yzx, zxy, zyx
    - reuse: xyx, xzx, yxy, yzy, zxz, zyz
    - can also use fixed axes, or rotated axes
    - that's 24 different versions!
- Quaternions



Quaternions Made Usable

- Discovered by Irish mathematician, Sir William Rowan Hamilton in 1843.
- Let rotationAxis = $[r_x, r_y, r_z]$, α = angle
- quaternion = [x, y, z, w], where

  $x = r_x * \sin(\alpha / 2)$

  $y = r_y * \sin(\alpha / 2)$

  $z = r_z * \sin(\alpha / 2)$

  $w = \cos(\alpha / 2)$

A little Quaternion algebra

- Lots of useful operators and operations

  Let q = [**v**, w], where **v**=[x,y,z]

  length(q) = |q| = sqrt(w^2 + **v•v'**)

  q+q' = [**v**+**v'**, w+w']

  q*q' = qq'= [**v**×**v'**+w**v'**+w'**v**, ww'-**v•v'**]

  where is × vector cross product

  Same as rotation composition (matrix multiplication)!

qq' ≠ q'q quaternion multiplication is **not** commutative

but rotations aren't either!

A few JOML Quaternion Methods

- Matrix* methods

```
set(Quaternionf q); // set mat to q rotation
rotate(Quaternionf q);// multiply by q rot.
```

- Quaternion* methods

```
rotationTo(Vector3f from, Vector3f to);
     // Quaternion that rotates 'from' -> 'to'
rotate( … ); // rotate this Quaternion by args
```

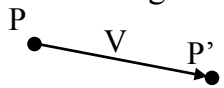

Quaternion Advantages

- Simple composition
- Obvious geometric interpretation (axis/angle)
  - simple to define high level rotations: map one direction to another
- Simple interpolation methods
- Compact representation (vs matrix)
- No *gimbal lock* (suffered by Euler angles)

Motivation for Quaternions

- Consider the relationship between a point, P and a vector, V:

- Can "apply" V to P to get a "new" P, P'



- Can apply a Quaternion to a Vector to get a "new" Vector:

- Vector has length and orientation; a Quaternion must represent a *relative* length and a *relative* orientation.
- A **Versor** is a quaternion of length 1; it is a pure rotation.

Overview of Quaternion Definition

- Motivated by complex numbers

  $c = a + b\boldsymbol{i}$ where $\boldsymbol{i} = sqrt(-1)$

  $q = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} + w$ where $\boldsymbol{i}=\boldsymbol{j}=\boldsymbol{k}= sqrt(-1)$
- Complex numbers can be modeled in 2D space

  x-axis is real part; *y*-axis is imaginary part
- Quaternions can be modeled in 4D

  xyz-axes is vector part; *w*-axis is angle part
- Restriction to unit length quaternions reduces (somewhat) the 4D complexity

Complex Number Reminders

- Complex number: $c = a + b\boldsymbol{i}$ where $\boldsymbol{i}^2=-1$

  $c_1+c_2 = (a_1+a_2) + (b_1+b_2)\boldsymbol{i}$

  $c_1*c_2 = c_1c_2 = (a_1+b_1\boldsymbol{i})(a_2+b_2\boldsymbol{i}) = a_1a_2+a_1b_2\boldsymbol{i}+a_2b_1\boldsymbol{i}-a_1b_2$

  $= (a_1a_2-b_1b_2) + (a_1a_2+b_1b_2)\boldsymbol{i}$

  $c^2 = (a^2-b^2) + 2ab\boldsymbol{i}$

  $Conjugate(c) = c^* = a-b\boldsymbol{i}$

  $norm(c)=length(c)=|c| = c\,c^* = sqrt(a^2+b^2)$

  $c_1/c_2 = (c_1c_2^*)/(c_2c_2^*)\boldsymbol{i} = (a_1a_2+b_1b_2)/(a_1^2+a_2^2)$

  $+(b_1a_2-a_1b_2)/(a_1^2+a_2^2)$

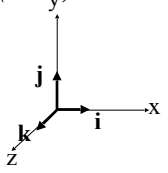
  and some more

Quaternion Basics

$q = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} + w$ where $\boldsymbol{i}=\boldsymbol{j}=\boldsymbol{k}= sqrt(-1)$

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$

$qq' = [\mathbf{v}\times\mathbf{v}'+w\mathbf{v}'+w'\mathbf{v},\ ww'-\mathbf{v}\bullet\mathbf{v}']$



Express $\boldsymbol{i}$ and $\boldsymbol{j}$ as *pure* quaternions (w=0)

$\boldsymbol{ij} = [\boldsymbol{i}\times\boldsymbol{j},\ -\boldsymbol{i}\bullet\boldsymbol{j}]$, $\boldsymbol{i}\bullet\boldsymbol{j}=\boldsymbol{0}$ and $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$

Similarly, $\boldsymbol{jk}=\boldsymbol{i}$ and $\boldsymbol{ki}=\boldsymbol{j}$

and $\boldsymbol{ji}=-\boldsymbol{k}$, $\boldsymbol{kj}=-\boldsymbol{i}$ and $\boldsymbol{ik}=-\boldsymbol{j}$ based on cross product

Quaternion Operations

- Addition/subtract: q+q' = [**v**+**v**', w+w'], q-q' = [**v**-**v**', w-w']
- Product: qq'= [**v**×**v'**+w**v'**+w'**v**, ww'-**v**•**v'**]

  qq'= [(wx'+w'x+yz'-y'z)***i***, 0] + [(wy'+w'y+zx'-z'x)***j***, 0]

  + [(wz'+w'z+xy'-x'y)***k***, 0] + [0,0,0], (ww'-xx'-yy'-zz')]
- Multiply by scalar: sq = [s**v**, sw]
- *real* quaternion: r=[(0,0,0), w] # no vector component
- *pure* quaternion: r=[**v**, 0] # 0 angle
- Conjugate, q*=[**v**,-w], so qq*= [$|v|^2$+w^2] and |q|=sqrt(qq*)
- Normalized quaternion = q/|q|
- Inverse: q^{-1}=q*/$|q|^2$
- Dot: q•q'=xx'+yy'+zz'+ww'=cos α, angle between q and q'
  if |q|=|q'|= 1