CS770/870 Spring 2017
3D Transformations

Related material

Any text on Computer Graphics

the Web

- 3D Transformations
  - Homogeneous coordinates
  - Matrices, vectors, points

3D Transformations

- 3D is (mostly) straightforward extension of 2D
- Homogeneous vector is 4x1
- Homogeneous matrix is 4x4
- Intuitive analogy is that a shear by w in 4D, results in a 3D translation in the w=1 plane

3D Homogeneous Scale Matrix

$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} S_x * x \\ S_y * y \\ S_z * z \\ 1 \end{bmatrix}$$

3D Homogeneous Translation Matrix

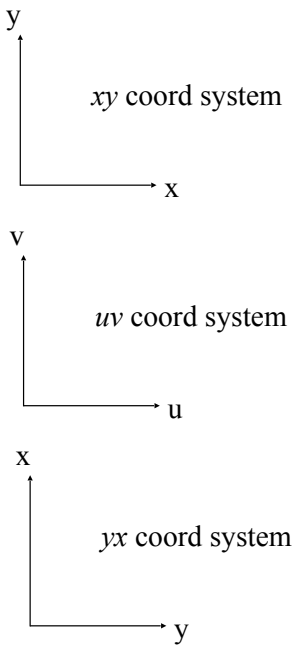
$$\begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} x + d_x \\ y + d_y \\ z + d_z \\ 1 \end{bmatrix}$$

3D Homogeneous Rotation

- 3D Rotation is more complicated
  - Rotation occurs about an axis, not a point
  - Need an unambiguous way to define the axis and to define what is a positive angle of rotation
  - This requires a careful definition of the coordinate system being used

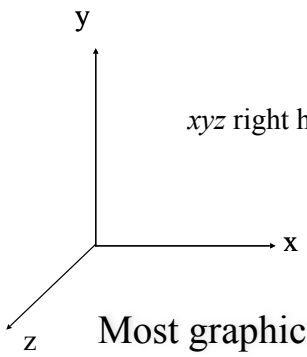
2D Axis Ordering Conventions

- In 2D
  - If we say the axes are labeled *xy,* we assume that *x* is horizontal, increasing to the right, and *y* is vertical increasing upward
  - It's pure convention!

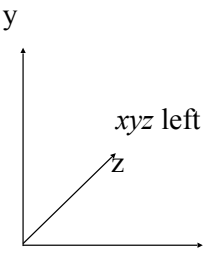


3D Axis Ordering Conventions

- Given *xyz* coordinate system, *xy* should match the 2D case with $z \perp xy$ plane
  - *z* can come toward the viewer (right-handed system)
    - thumb is *x,* index finger is *y,* big finger is *z*
  - *z* can go away from the viewer (left-handed system)



xyz right handed system



xyz left handed system

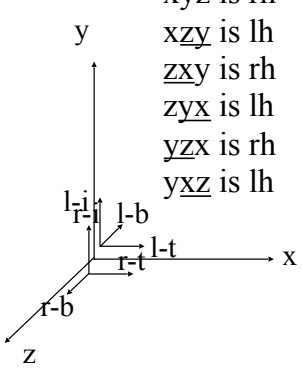
Most graphics system software defaults to right handed.

It's a labeling convention!

- The same axis labeling can be listed in any order
  - Circular rotation of the same letters has no effect

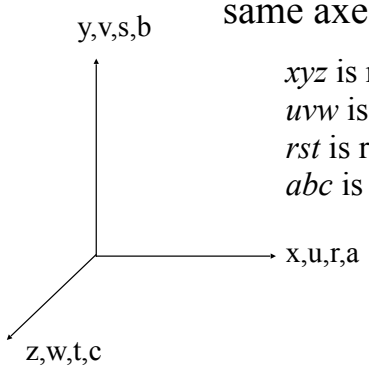
    xyz, zxy, yzx are all rh
  - Swapping 2 adjacent letters changes handedness

xyz is rh
xzy is lh
zxy is rh
zyx is lh
yzx is rh
yxz is lh



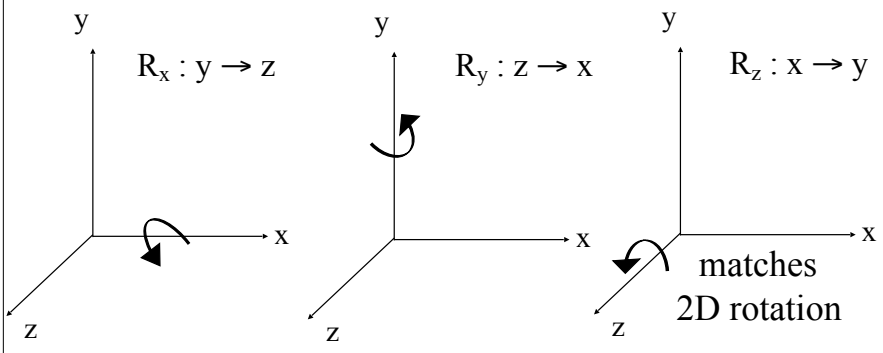
Any labeling can be assigned to the same axes.

xyz is rh
uvw is rh
rst is rh
abc is rh



3D Rotation Conventions

- Rotation is defined about an *axis*
- Elementary rotations are defined about each of the principal coordinate axes
  - A positive angle is *CCW* as seen by looking at the origin from $+\infty$ along the axis of rotation



mnemonic:
given xyz coord
$x \rightarrow y \rightarrow z \rightarrow x$
$R_z \quad R_x \quad R_y$

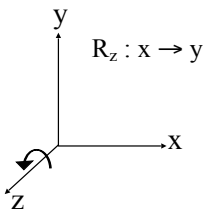
3D Homogeneous Rotation Matrices

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_y = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

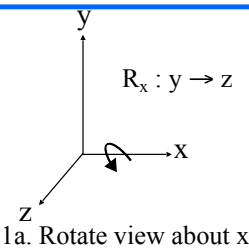
Map Rotations to Same View

Canonical representation:
xyz axes: rotate about z

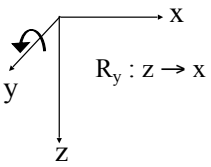


Can re-orient the axes to show that all RH rotations are really same:

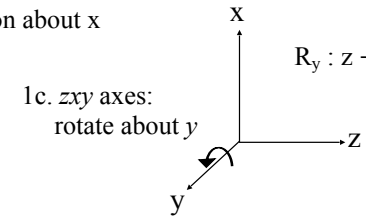
1. Map to rotation about y
2. Map to rotation about x

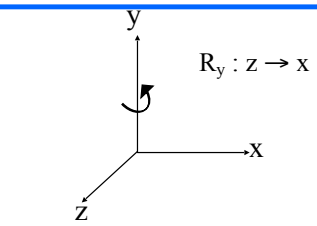


1a. Rotate view about x, get y out

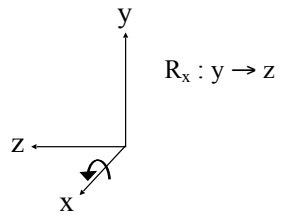


1b. Rotate view about y, get x up

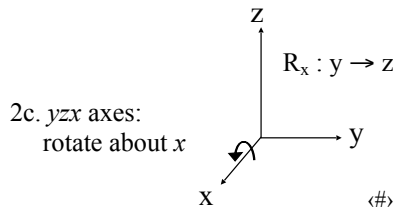




2a. Rotate 270° about y, get x out

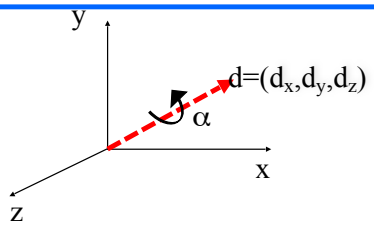


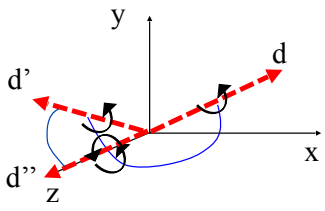
2b. Rotate 270° about x, get z up



Rotation about Arbitrary Direction

- Rotate α degrees about a *direction,* $d=(d_x,d_y,d_z)$ using axis through (0,0,0)
  1. R_y: rotate world about y-axis until transformed d is in YZ
  2. R_x: rotate world about x-axis until d' along z
  3. **Rotate α about z**
  4. "Undo" R_x
  5. "Undo" R_y





Next: do this in more detail

Rotation about Arbitrary Axis

- Rotate α degrees about an axis defined by a point (c_x,c_y,c_z) and a *direction,* $d=(d_x,d_y,d_z)$
  - Translate (c_x,c_y,c_z) to origin: $T=T(-c_x,-c_y,-c_z)$

Key steps

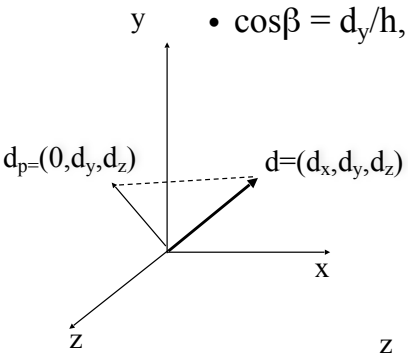
  - Rotate about x so that (d_x,d_y,d_z) lies in xy plane
  - Rotate about z so that d' (transformed d) lies along y axis
  - Rotate α degrees about y axis: $R_y(\alpha)$
  - Apply inverse of z rotation
  - Apply inverse of x rotation
  - Apply inverse of translation: $T^{-1}=T(c_x,c_y,c_z)$

Composite transformation: $R = T^{-1}R_x^{-1}R_z^{-1}R_y(\alpha)R_zR_xT$

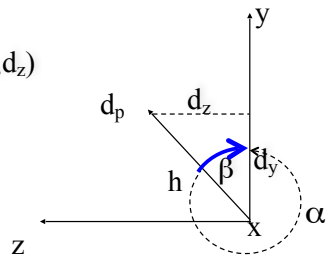
R_x: map d to xy plane

R_x to get $d=(d_x,d_y,d_z)$ into xy plane

- Project d onto yz plane: $d_p=(0,d_y,d_z)$
- Rotate about x to map d_p to y
- Since rotating z to y, $\beta < 0$, need $\alpha = 360-\beta$
- Don't need β, but cos β and sin β
- $\cos\beta = d_y/h$, $\sin\beta = d_z/h$ where $h = \sqrt{d_y*d_y + d_z*d_z}$



3D sketch



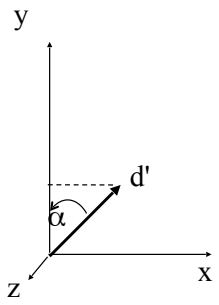
2D sketch

- $\alpha=360-\beta$
- $\cos(360-\beta) = \cos\beta$
- $\sin(360-\beta) = -\sin\beta$

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d_y/h & d_z/h & 0 \\ 0 & -d_z/h & d_y/h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R_z: Map d' to y axis

- d' = R_x*d = (d'$_x$,d'$_y$,0)
- L = sqrt(d_x*d_x + d_y*d_y + d_z*d_z)=length(d)=length(d')
- cosα = d'$_y$/L
- sinα= d'$_x$/L



$$R_z = \begin{bmatrix} d'_y/L & -d'_x/L & 0 & 0 \\ d'_x/L & d'_y/L & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

• Note: if d is a *unit* vector (or has been converted to one), L = 1.

Composite Rotation Matrix

- The composite rotation matrix is:

$$R = T(c_x,c_y,c_z)R_x^{-1}R_z^{-1}R_y(\alpha)R_zR_xT(-c_x,-c_y,-c_z)$$

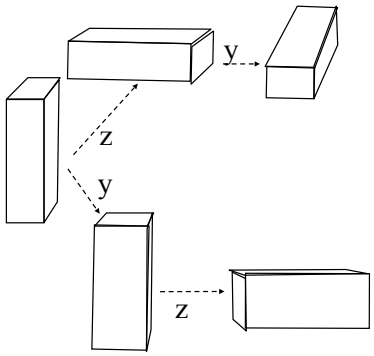
  - but what are the inverse rotations?
  - $R^{-1}(\alpha) = R(-\alpha)=R^T(\alpha)$ = transpose of $R(\alpha)$
  - And
    - cos α = cos(-α)
    - sin α = -sin(-α)
  - So, just change the location of the -sin term
- A lot of computation, but composite is computed once and applied to hundreds or thousands of points.

Alternate Rotation Formulation

- We arbitrarily chose to use R_x to map to xy plane, then R_z to align with y-axis
- Could do R_x, then R_z to align with x-axis
- Could do R_x to map to xz plane then R_z to align with x-axis, then R_y to align with either x- or z-axis
- Do some!

Are 3D Rotations Symmetric?

- Given two affine transforms, M_1 and M_2
  - Does $M_1 * M_2 = M_2 * M_1$?
  - try 90° rotation about y and z.

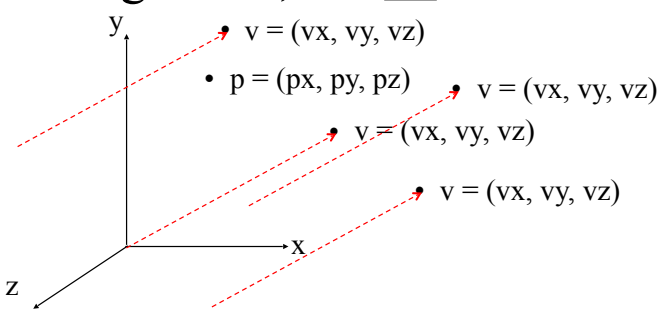


Rotation Matrices

- Pure rotation matrices are orthogonal
  - All columns are unit length
  - All columns are mutually orthogonal
    - their dot products are 0
- All the basic rotations are orthogonal
- Any composition of orthogonal matrices is orthogonal
- $R^{-1}=R^{T}$ for any orthogonal matrix R

Points and Vectors

- 3D points and vectors both need 3 components.
  - Point: represents a location in a coordinate system; it has no magnitude and no direction
  - Vector: represents a direction in a coordinate system; it has a magnitude, but no location



Homogeneous Coordinate Vector

- Homogenous coordinates provide a convenient opportunity to disambiguate the Vector/Point representation:
  - 3D point: [x y z 1]
  - 3D vector [x y z 0]
    - Matrix * vector operation "ignores" translation part of the Matrix -- since those parts are multiplied by 0
    - But, that is appropriate: vectors have no location (or they are at all locations), so translation is meaningless

Coordinate Systems

- A 3D coordinate system requires specification of an *origin* plus 3 orthogonal *vectors* representing the coordinate system *axes*
  - but, what coordinate system are those defined in?
  - usually a coordinate system's origin is *implicit* and treated as (0,0,0) and its vectors are (1,0,0), (0,1,0) and (0,0,1) and we just accept it as given
- What happens when we need to deal with multiple coordinate systems simultaneously?
  - need to be more rigorous!

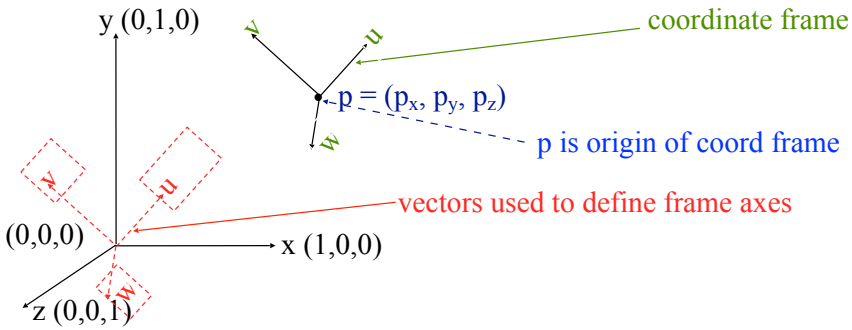
Coordinate Frame

- A coordinate *frame* explicitly describes a coordinate system in another coordinate system via 3 orthogonal vectors and a point:
  F={u,v,w,p}
- Can compactly represent a coordinate frame as a 4x4 matrix:

$$F = \begin{bmatrix} u_x & v_x & w_x & p_x \\ u_y & v_y & w_y & p_y \\ u_z & v_z & w_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Coordinate Frame Example

- Normally, specify coordinate frames in a *world* coordinate system, whose origin is (0,0,0) and whose axes are defined by (1,0,0), (0,1,0), and (0,0,1)



Coordinate Frame Transformation

- Consider the Frame matrix as a transformation
  - What does it do to the frame's u-axis vector ([1 0 0 0])?

$$F * \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}u_x & v_x & w_x & p_x\\ u_y & v_y & w_y & p_y\\ u_z & v_z & w_z & p_z\\ 0 & 0 & 0 & 1\end{bmatrix} * \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}u_x\\u_y\\u_z\\0\end{bmatrix}$$

  - It transforms it to the world's representation
  - Similarly, the v-axis [0 1 0 0]→[v_x v_y v_z 0]
    and the w-axis [0 0 1 0]→[w_x w_y w_z 0]
  - Finally, the frame origin [0 0 0 1]→[p_x p_y p_z 1]
  - This works for <u>any</u> point represented in the frame

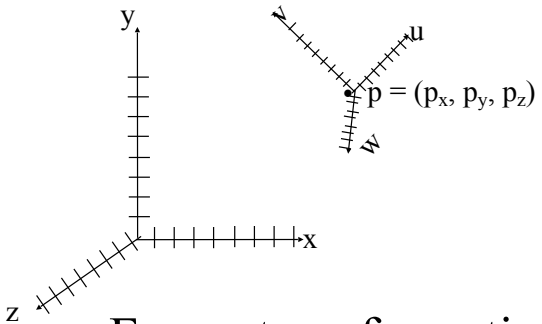
Compositing Frame Transformations

- Let F_1 be a frame transform specified in the *world*.
- Let F_2 be a frame transform specified in F_1 coords.
- How do we map coordinates in F_2 to world?
  - F_2 matrix maps from F_2 coords to F_1 coords
  - F_1 matrix maps from F_1 coords to world coords
  - So F_1*F_2 will map any coordinate from F_2 to world
- Key: successive frame transformations are composed by *post-multiplying* the new transform times the previous.
  - This composition is opposite to the ordering of successive transformations of the points, but the semantics is consistent.

Inverse Transformation

- Sometimes we have points in the world that need to be represented in a coordinate frame.
  - Can get these by finding the inverse of the coordinate frame transformation
  - General inverse matrix calculations are expensive
  - Composite affine transforms built from simple components are cheap:
    - $T^{-1}(d_x, d_y, d_z) = T(-d_x, -d_y, -d_z)$
    - $S^{-1}(s_x, s_y, s_z) = S(1/s_x, 1/s_y, 1/s_z)$
    - $R^{-1}(\alpha) = R(-\alpha) = R^T(\alpha)$ -- transpose matrix!
      - In fact, the inverse of any composition of rotation matrices is the transpose of the composite!

Inverse Transformation Example



uvw frame specification:
- Composite rotation, R
- Scale, S(s_x, s_y, s_z)
- Translate(p_x, p_y, p_z)

- Frame transformation, F = TRS
- $F^{-1} = (TRS)^{-1} = S^{-1}R^{-1}T^{-1}$

  $= S(1/s_x, 1/s_y, 1/s_z)\ R^T\ T(-p_x, -p_y, -p_z)$

Frame Transformation Revisited

- F defined by vectors defining the new axes
- These vectors need not be unit length!
- How are scale and rotate embedded in the upper 3x3 submatrix?

$$F = \begin{bmatrix} u_x & v_x & w_x & p_x \\ u_y & v_y & w_y & p_y \\ u_z & v_z & w_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $A = U/\|U\|$, $B = V/\|V\|$, $C = W/\|W\|$ (Unit length vectors)

Let $R = \begin{bmatrix} a_x & b_x & c_x & 0 \\ a_y & b_y & c_y & 0 \\ a_z & b_z & c_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} \|U\| & 0 & 0 & 0 \\ 0 & \|V\| & 0 & 0 \\ 0 & 0 & \|W\| & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

It's easy to show that $F=TRS$

Hierarchical Object Definition

- A *Shape* class object definition should define an object in a convenient "local" coordinate system
- Instances of the *Shape* class should be placed into another coordinate system subject to an appropriate transformation.
  - could be the "world" or another *Shape's* local coordinate system.
- Results in *nested* hierarchical shape definition

Review

- Coordinate systems
- 2D Transformations
- Homogeneous coordinates
- Matrices, vectors, points
- 3D Transformations
- Coordinate Frames

Next

- 3D Viewing
- Synthetic camera model
- Modeling transformations
- Viewing transformations