CS770/870 Spring 2017 Transformations

Coordinate systems

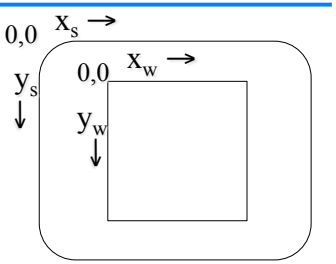
2D Transformations

Homogeneous coordinates

Matrices, vectors, points

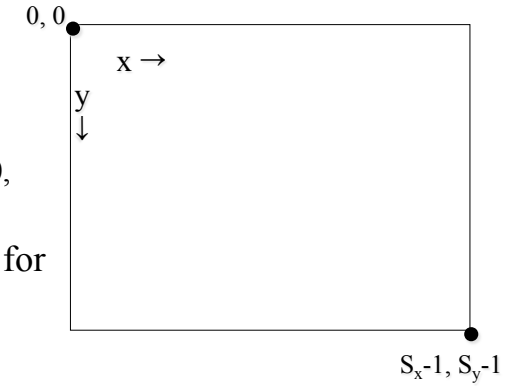
Coordinate Systems

- Coordinate systems used in graphics
  - *Screen* coordinates: the physical location on a specific display device
  - *Window* coordinates: screen coords relative to window origin
  - *Normalized device coordinates (NDC)*: a device independent abstraction of display window
  - *World coordinates*: a generic coordinate system that any application can use to define objects for display



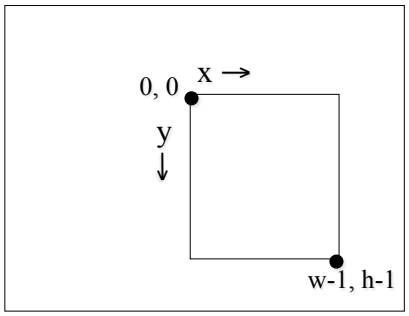
Screen Coordinates

- Physical device coords
  - Location of pixels
  - S_x by S_y pixels
    - 640x480, 1024x768, 1280x800, 1440x900, etc.
  - Variation is difficult for application program; leads to device dependent code



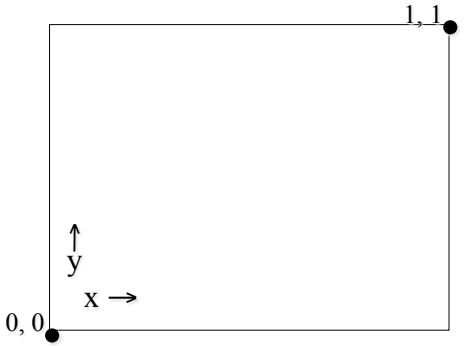
GUI Window Coordinates

- GUI windows add another complication
  - GUI Window system gives the application a portion of the screen that is *w* x *h* pixels, for arbitrary *w* and *h*
  - Window system maps window coordinates to screen coordinates



Normalized Device Coordinates

- An *abstraction* of display device coordinates (screen and/or GUI window)
- Floating point (x,y) coordinates from 0 to 1
- y goes up!



World Coordinates

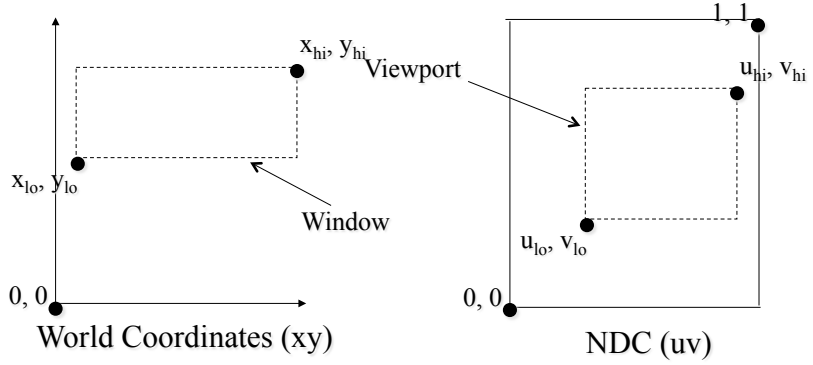
- The application should be able to define objects in a way natural for the domain:
  - 0 to 100 feet for a house floor plan
  - 0 to 4000 miles for a map of the US
  - etc.
- *World coordinates* are simply 2D or 3D *float* or *double* values that can represent locations and sizes of object to be displayed

Mapping World Coordinates to Device Coordinates

- Application needs to tell the graphics system:
  - what portion of the world is to be displayed, called the *window* on the world.
  - where on the screen the window should be mapped, called the *viewport*.
- In 2D
  - the window is a rectangle in world coordinates
  - the viewport is a rectangle in normalized device coordinates
  - Some graphics systems allow a rotation as well

Window to Viewport Mapping

- Window: (x_{lo}, y_{lo}) to (x_{hi}, y_{hi})
- Viewport: (u_{lo}, v_{lo}) to (u_{hi}, v_{hi})

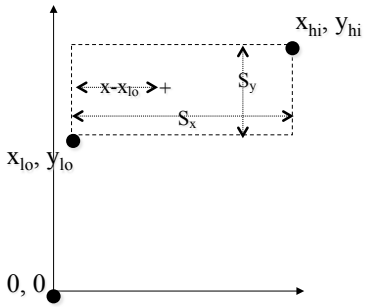


Window to Viewport Mapping

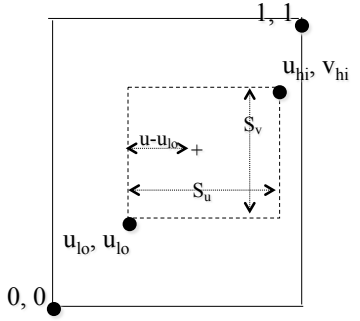
- Given (x,y) in world, what is its (u,v) in ndc?
  - Consider the x coordinate first
    - Let S_x be the width of the window = $x_{hi} - x_{lo}$
    - Let S_u be the width of the viewport = $u_{hi} - u_{lo}$
    - what % of the way from x_{lo} to x_{hi} is x?
      xratio = $(x - x_{lo}) / S_x$
    - it is exactly that % of the way from u_{lo} to u_{hi} in ndc:
      uratio = $(u - u_{lo}) / S_u$
    - so, $(u - u_{lo}) / S_u = (x - x_{lo}) / S_x$
    - or, $u = u_{lo} + (x - x_{lo}) / S_x * S_u$
  - Similarly, $v = v_{lo} + (y - y_{lo}) / S_y * S_v$

Window to Viewport Mapping 2

- $u = u_{lo} + (x - x_{lo}) / S_x * S_u$
- $v = v_{lo} + (y - y_{lo}) / S_y * S_v$



World Coordinates



NDC

Clipping

- What if (x,y) is outside the window?
  - it should not be visible in the viewport!
  - can't just ignore a point outside window; it's part of a line or polygon or more complex object
  - must display only the portion of each object that is inside the window
  - must do it efficiently!
  - we'll re-visit clipping later

2D Transformations

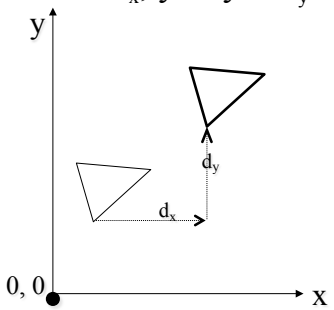
- We'd like to *define* an object with 1 set of coordinate specifications, then *transform* it to another location, size, and/or orientation.
- Translation by (d_x, d_y)
- Scale by (s_x, s_y)
- Rotation by α about the origin (0,0)

2D Translation

- Translation by (d_x, d_y)

  for each (x,y) in the object

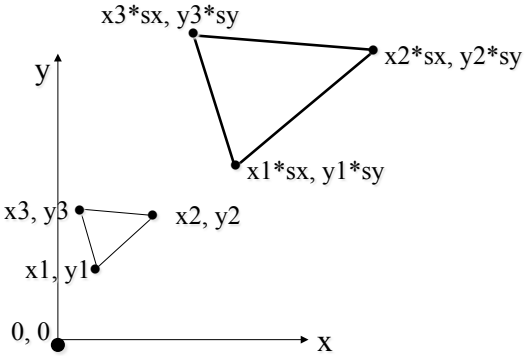
  $x' = x + d_x$; $y' = y + d_y$



2D Scale

for each (x,y) in the object

$x' = x * sx$; $y' = y * sy$



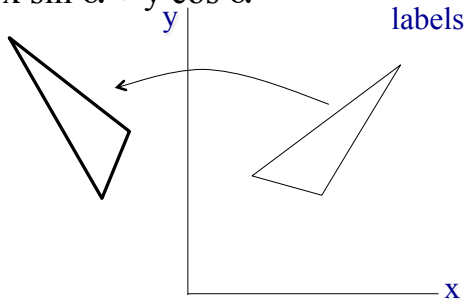
2D Rotation

- Rotation of α degrees about the origin, where $\alpha > 0$ is counter clockwise rotation

$x' = x \cos \alpha - y \sin \alpha$

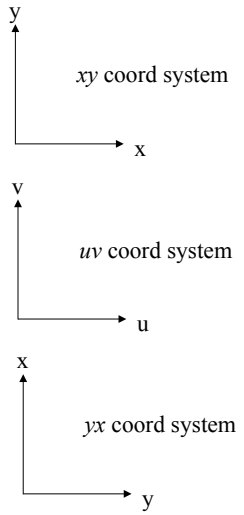
$y' = x \sin \alpha + y \cos \alpha$

Note: equations only make sense if the axes labels are known!



2D Axis Ordering Conventions

- If we say the axes are labeled *xy,* we assume that *x* is horizontal, increasing to the right, and *y* is vertical increasing upward
  - It's pure convention!
- What is a positive angle?
  - if axes are *xy*, an angle >0 rotates *x*-axis towards *y*
  - another convention.



Affine Transformations

- Affine transformations leave the origin (0,0) unchanged
  - Scale and rotation are affine transformations
  - Translation is not affine
- 2D affine transformations can be elegantly represented in matrix format as 2x2 matrices
  - unifies the representation of scale and rotation
  - provides an easy implementation of composition of transformations: rotate, then scale, then rotate

Linear Algebra

- Linear algebra isn't scary: it's notation. Consider the 2 linear equations on the right
- The "meaning" of the equations is all in the constants (a,b,c,d); the variables (u,v,x,y) are just placeholders. We encapsulate the constants in a 2x2 **matrix** and the variables as 2x1 **vectors**.

$$u = ax + by$$

$$v = cx + dy$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear Algebra Operators

- To utilize this "notation" efficiently, we can define some operators on matrices and vectors:
  - "Multiplication" of a matrix times a vector let's us recover the underlying equations
  - The "dot" product (also called *inner* product) let's us combine 2 vectors into a scalar; it's definition leads to convenient computations of vector lengths and the angle between two vectors
  - The "cross" product (*outer* product) let's us easily compute a new vector that is orthogonal to 2 vectors

Dot (inner) Product

- Given two vectors of the same length, the dot product is the sum of the products of corresponding entries:

$$\begin{bmatrix} a \\ b \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \text{ax+by}$$

which really should be written as

a row vector * a column vector

$$\begin{bmatrix} a & b \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \text{ax+by}$$

Matrix times a vector

- Consider the 2x2 matrix and a 2x1 vector:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}$$

- The product of the matrix * vector is defined as

$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

  - Each entry is the dot product of a row of the input matrix with the input vector (thus the row*col form of the dot product definition)

Matrix times Matrix

- The row/column dot product convention extends to matrix products:
- Row i of the left operand is dotted with column j of the right operand to get position (i,j) in the result:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} r & s \\ t & u \end{bmatrix} \rightarrow \begin{bmatrix} ar + bt & as + bu \\ cr + dt & cs + du \end{bmatrix}$$

- All this extends easily to 3D and more

Matrices for Affine Transforms

- Scale: $x' = xS_x$, $y' = yS_y$

$$\begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

- Rotation: $x' = x \cos\alpha - y \sin\alpha$, $y' = x \sin\alpha + y \cos\alpha$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Assumes axes are labeled *xy*.
+90 rotation: cos 90 = 0, sin 90 = 1

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

x axis (1,0) ⇒ y axis (0,1)
y axis (0,1) ⇒ -x axis (-1,0)

Matrix Multiplication

- Matrix multiplication is just a series of matrix-vector multiplications
  - Each column of the right hand matrix is multiplied by the left to get the next column of result

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} u & x \\ w & y \end{bmatrix} \rightarrow \begin{bmatrix} au + bw & ax + by \\ cu + dw & cx + dy \end{bmatrix}$$

  - Matrix multiplication is <u>associative</u>
    (M1 * (M2 * V)) = (M1 * M2) * V

Transformation Composition

- Suppose you want to scale an object by (s, t), then rotate by α degrees
  - Create S and R matrices
  - Let M = R*S
  - foreach point in object, p
    newP = M * p
- Simpler and more efficient than first transforming all points by S and then transforming the results by R. Why?

Are Affine Transforms Symmetric?

- Given two affine transforms, M_1 and M_2
  - Does $M_1 * M_2 = M_2 * M_1$?
  - Consider S = scale by (1, 2), R = rotate by 90
    - Does S*R = R*S
    - Surely, not.
  - What about 2 rotations?
    - yes (in 2D, but not 3D)
  - What about 2 scales?
    - yes
  - Rotation and uniform scale?
    - yes

THE Problem

- Translation is <u>not</u> affine
  - but we can't live without it!
- How can we combine translation with scale and rotate into one unified model?
- Homogeneous coordinates
  - We'll present it as a convenient engineering "trick"
  - There is a rigorous mathematical explanation

Translation Transformation

- $x' = x + d_x, \quad y' = x + d_y$
- Consider the 3x3 matrix and the 3x1 vector:

$$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- The product of this matrix times the vector is:

$$\begin{bmatrix} x + d_x \\ y + d_y \\ 1 \end{bmatrix}$$

It's just what we want (if we ignore the extra coord)!

But! what about scale and rotation?

Revised Scale/Rotate Matrices

- Expand the Scale and Rotate matrices to 3x3

$$\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

You should verify that multiplying these matrices by [x y 1] produce same results as before.

Homogeneous Coordinates

- The 3x3 matrix representation for 2D transformations is called *homogeneous coordinates*
- The 3rd coordinate is called the homogeneous coordinate and is usually designated with a *w*
- In this model, rotation, scale and translation can be represented in a single uniform notation and can be composed in arbitrary order

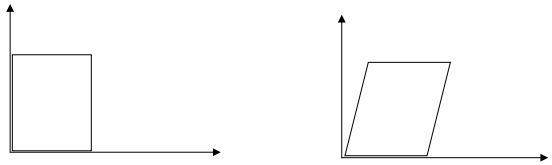
Why does it work?

- There is a complex mathematical explanation and
- an intuitive geometric one.

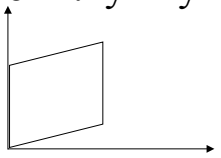
Which do you want?

2D Shear Transform

- Shear of x by a factor of y: $x' = x + y*h_{xy}$
  - turns a rectangle into a parallelogram

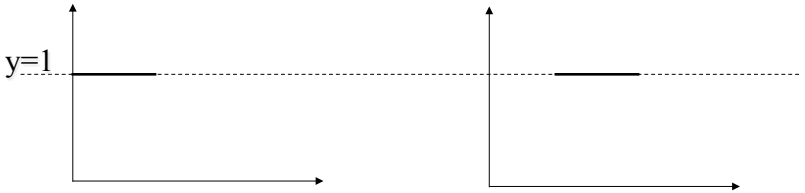


- Shear of y by a factor of x: $y' = y + x* h_{yx}$
  - similar result



What is a 2D shear in 1D?

- Suppose you represent all your objects with y=1; this becomes a 1D world with scale and translation operations.
- A 2D shear becomes a 1D translation at y=1



- Similarly, a 3D shear at w=1 is a 2D translation in the w=1 plane

2D Shear Matrix

- 2x2 shear matrix

$$\begin{bmatrix} 1 & h_{xy} \\ h_{yx} & 1 \end{bmatrix}$$

- 3x3 homogeneous shear matrix for 2D shear

$$\begin{bmatrix} 1 & h_{xy} & 0 \\ h_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify that these are correct

3D Shear Matrix

- A complete 3D shear matrix is

$$\begin{bmatrix} 1 & h_{xy} & h_{xz} \\ h_{yx} & 1 & h_{yz} \\ h_{zx} & h_{zy} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & h_{xw} \\ 0 & 1 & h_{yw} \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

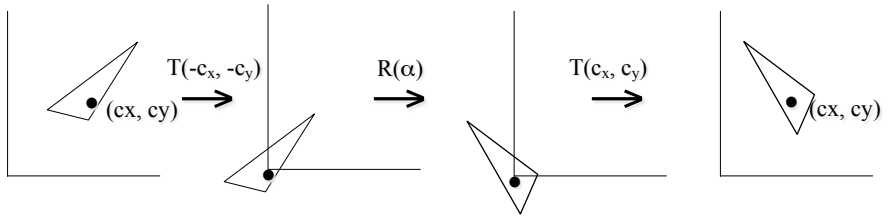
- But, when we use the matrix as a 2D homogeneous matrix, all the factors except h_{xz} and h_{yz} are 0
  - and those should be renamed as h_{xw} and h_{yw}
  - or even d_x, d_y

Scale/Rotation about a Point

- Scale/Rotation leave the origin fixed
  - we say that they occur "about the origin"
- Often, want the transformation to occur about an arbitrary point, perhaps the object's center
- Can do this with a composite transformation that:
  - translates the desired center point to the origin
  - applies the scale or rotate (or both)
  - translates the origin back to the original center point
- The composite is applied to all points

Arbitrary Rotation

- Rotate by α about (c_x, c_y)
  - T(-c_x, -c_y): Translate (c_x, c_y) to (0, 0)
  - R(α): Rotation by α
  - T(c_x, c_y): Translate (0, 0) to (c_x, c_y)



Arbitrary Rotation Matrices

- Rotate by α about (c_x, c_y)
  - T_1, Translate (c_x, c_y) to (0, 0) $T_1 = \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix}$
  - R, Rotation $R = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  - T_2, Translate (0, 0) to (c_x, c_y) $T_2 = \begin{bmatrix} 1 & 0 & c_x \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{bmatrix}$
- Composite $C = T_2 R T_1$
  - for each point P: $P' = CP$

Multiplication Exercise

- $C=T_2RT_1$

1. T_2*R
   r_0*c_0
   r_0*c_1
   r_0*c_2
   etc
2. (T_2R)*T_1

$$C = \begin{bmatrix} 1 & 0 & c_x \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & c_x \\ \sin\alpha & \cos\alpha & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -c_x \\ 0 & 1 & -c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & -c_x\cos\alpha + c_y\sin\alpha + c_x \\ \sin\alpha & \cos\alpha & -c_x\sin\alpha - c_y\cos\alpha + c_y \\ 0 & 0 & 1 \end{bmatrix}$$

Review/Preview

- Review
  - Coordinate systems
  - 2D Transformations
  - Homogeneous coordinates
  - Matrices, vectors, points
- Preview
  - 3D Transformations
  - Homogeneous coords
  - Matrices, vectors, points